Yiddish of the Day

"fartik the fish" = פֿאַרטיק די פֿיש ...

the fish is ready =

"After a few months, though, I realized something: I hadn't gotten any better at understanding tensor-products, but I was getting used to not understanding them. It was pretty amazing, I no longer felt anguish when tensor products came up; I was instead amused by their cunning ways"

— Cathy O'neil

"It is the things you can prove that tell you how to think about tensor products. Ie, you let lemmas and examples shape your intuition of the mathematical objects in question. There's nothing else, no magical intuition will appear to help you understand it"

— Johan de Jong

Multilinear Algebra

- **Def:** Let V_1, V_2, W be vector spaces. Then a <u>bilinear map</u> is a function

$$f: \underline{V_1 \times V_2} \longrightarrow \underline{W}$$

such that

1) $f(v_1 + v_1', v_2) = f(v_1, v_2) + f(v_1', v_2)$

2) $f(v_1, v_2 + v_2') = f(v_1, v_2) + f(v_1, v_2')$

3) $r f(v_1, v_2) = f(r v_1, v_2) = f(v_1, r v_2)$

ii) More generally if $V_1 \dots V_n$, W are vector spaces, then a **multilinear map** is a function

$$f: \underline{V_1 \times V_2 \times \dots \times V_n} \longrightarrow \underline{W}$$

st 1) $f(v_1 + v_1', \dots, v_n) = f(v_1, \dots, v_n) + f(v_1', \dots, v_n)$

$\vdots \qquad \vdots$

n) $f(v_1, \dots, v_n + v_n') = f(v_1 \dots v_n) + f(v_1 \dots, v_n')$

n+1) $r f(v_1 \dots v_n) = f(rv_1, \dots, v_n) = f(v_1, rv_2, \dots, v_n)$

$= \dots = f(v_1 \dots, rv_n)$

Rmk: When $n=1$ $\longrightarrow$ linear maps

$n=2$ $\longrightarrow$ bi-linear maps

iii) When $W=\mathbb{F}$ and V_i are same

$n=1$: linear functional

$n=2$ called "bilinear form"

$n>2$: called "multi-linear forms"

Prop: The set $\mathrm{Mult}(V_1 \times \cdots \times V_n, W)$ of multilinear maps is a vector space under pointwise addition/scaling

Pf) HW (n=2 case)

(no kernel, image not subspace, etc...)

Goal: 1) Multilinear maps = ☹ booooo!

Linear maps = ☺ wooooh!

→ Want to convert multilinear → linear maps

2) Want some way to "multiply" vectors

3) We know that a vector space V over $\mathbb{R}$, isn't always a vector space over $\mathbb{C}$. But $\mathbb{C}$ is better than $\mathbb{R}$, can we make it into a $\mathbb{C}$ vector space?

Deep Breath

Goal: Want a bijection

$$\mathrm{Bilinear}(V_1 \times V_2, W) \simeq \mathcal{L}(\ ?\ , W)$$

- What should this ? be

General construction

Let V_1, V_2, W be vector spaces.

1) Consider the **free** vector space $F^{(V_1 \times V_2)}$

(see notes)

rmk: Given a function $f: V_1 \times V_2 \to W$ we can uniquely extend this to a linear map

$$F^{(V_1 \times V_2)} \longrightarrow W$$

(why?): Because $V_1 \times V_2$ is a basis for $F^{(V_1 \times V_2)}$

2) Consider this weird subspace

$$N : \left\langle \begin{array}{l} (v_1+v_1', v_2) - (v_1, v_2) - (v_1', v_2), \\ (v_1, v_2+v_2') - (v_1, v_2) - (v_1, v_2'), \\ (rv_1, v_2) - (v_1, rv_2) \end{array} \middle| \begin{array}{l} v_1, v_1' \in V_1 \\ v_2 \in V_2 \\ v_2' \in V_2 \end{array} \right\rangle$$

Why?: If $f: V_1 \times V_2 \to W$ is bilinear

then have induced linear map

$\mathrm{Free}(f) : F^{(V_1 \times V_2)} \longrightarrow W$ as before.

Note: $N \subseteq \mathrm{Ker}(\mathrm{Free}(f))$

3) Consider $F^{(V_1 \times V_2)} / N$

Why: Given bilinear $f: V_1 \times V_2 \to W$

we saw we had an induced map

$\mathrm{Free}(f): F^{(V_1 \times V_2)} \to W$

with $N \subseteq \mathrm{Ker}(\mathrm{Free}(f))$

$\Rightarrow$ Universal property of quotient

tells us that this map can

then be extended to a unique linear map

$\widetilde{\text{Free}(f)} : \underline{F^{(V_1 \times V_2)}/N} \longrightarrow W$

$$\boxed{\underline{\text{Define}}: \ \underline{V_1 \otimes V_2} := F^{(V_1 \times V_2)}/N}$$

Step back: What have we acheived?

- Start with a <u>bilinear</u> map $f : \underline{V_1 \times V_2} \longrightarrow W$

Then we have the following diagram

$$\begin{tikzcd}
\underline{V_1 \times V_2} \arrow[r, "f"] \arrow[d, "i"] & W \\
F^{(V_1 \times V_2)} \arrow[ur, "\exists!\, \mathrm{Free}(f)", "(1)"'] \arrow[d, "\pi_N"] & \\
\underline{V_1 \otimes V_2} =: F^{(V_1 \times V_2)}/N \arrow[uur, dashed, "\exists!\, \tilde{f}"', "(2)"] &
\end{tikzcd}$$

$(f(v_1, v_2) = \mathrm{Free}f(v_1, v_2))$

1) $f = \mathrm{Free}(f) \circ i$

2) $\mathrm{Free}(f) = \tilde{f} \circ \pi$

$$\Rightarrow f(v_1, v_2) = \tilde{f} \circ \pi \circ i\,(v_1, v_2)$$
$$= \tilde{f}(v_1 \otimes v_2)$$

Thm: The composite $\underline{V_1 \times V_2} \hookrightarrow F^{(V_1 \times V_2)} \to \underline{V_1 \otimes V_2}$ is bilinear and it induces a (natural) isomorphism

$$\mathrm{Bil}(\underline{V_1 \times V_2}, W) \simeq \mathcal{L}(V_1 \otimes V_2, W)$$

(This is the so called "universal" property of the <u>tensor-product</u>)

- We write $\pi : \underline{V_1 \times V_2} \longrightarrow \underline{V_1 \otimes V_2}$ to be this <u>bilinear</u> map and denote $\underline{a \otimes b} := \pi(a, b)$

- <u>Rmk</u>: Every vector in $\underline{V_1 \otimes V_2}$ is a finite sum of these "$\underline{\text{simple tensors}}$" $\left(\sum c_{ij}\, a_i \otimes b_j\right)$

- Because $\underline{a \otimes b}$ is really $\pi(a, b)$

  we have things are "linear" in each slot

  ex) i) $0 \otimes a = (0+0) \otimes a = 0 \otimes a + 0 \otimes a$

  $\Rightarrow 0 \otimes a = 0$

  ii) $(a+b) \otimes c = a \otimes c + b \otimes c$

Same exact construction holds for $V_1, \ldots, V_n, W$

and <u>multi-linear</u> maps

Thm: The composite $\underline{V_1 \times \cdots \times V_n} \hookrightarrow F^{(V_1 \times \cdots \times V_n)} \longrightarrow F^{(V_1 \times \cdots \times V_n)}/N = V_1 \otimes \cdots \otimes V_n$ is **multi-linear** and induces a bijection

$$\mathrm{Mult}(V_1 \times \cdots \times V_n, W) \simeq \mathcal{L}(V_1 \otimes \cdots \otimes V_n, W)$$

$$\begin{tikzcd}
\underline{V_1 \times \cdots \times V_n} \arrow[r, "f"] \arrow[d, "\pi"] & W \\
\underline{V_1 \otimes \cdots \otimes V_n} \arrow[ur, dashed, "\exists ! \tilde{f}"'] &
\end{tikzcd}$$

$\tilde{f}(a_1 \otimes \cdots \otimes a_n) = f(a_1, \ldots, a_n)$

Obs: The machinery developed above is very general, and therefore very powerful.

However, in the world of vector spaces, we have

basis !

This simplifies things!!!

Thm: Let V, W be vector spaces with basis

$B_V = (v_1 \dots v_n)$ $B_W = (w_1 \dots w_m)$.

Then $V \otimes W$ has basis

$$\underline{B_{V\otimes W}} = \left(v_i \otimes w_j \;\middle|\; \begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix} \right)$$

$\underline{\text{Cor}}$: $\dim(V \otimes W) = \underline{\dim V \dim W} = nm$

(Compare again to $\underline{\dim(V \oplus W) = n+m}$)

- We will use this to compute matrix of tensor product of linear maps.

Suppose $T_1 : V_1 \to W_1$, $T_2 : V_2 \to W_2$ are linear maps between $\mathbb{F}$-vector spaces V_1, V_2, W_1, W_2

Prop: There exists a unique linear map

$$T_1 \otimes T_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

such that $T_1 \otimes T_2 (v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$

Pf) HW :) (Hint: Universal property of $\otimes$:))

⌜Define the map

$$\begin{aligned} V_1 \times V_2 &\longrightarrow W_1 \times W_2 \longrightarrow W_1 \otimes W_2 \\ (v_1, v_2) &\longrightarrow (T(v_1), T(v_2)) \longrightarrow T(v_1) \otimes T(v_2) \end{aligned}$$

Check the composite is bilinear ⌟

Special Cases

$V_1 \xrightarrow{id} V_1$

Consider a map $V \xrightarrow{T} W$. Then for vector space V' get ! map

$$V' \otimes V \xrightarrow{1 \otimes T} V' \otimes W$$

a) Let $V = W$ and $f = id$. Then what is this map

$$V' \otimes V \xrightarrow{1 \otimes 1} V' \otimes V$$

(ie when both T_1, T_2 are the identity maps)

$$(1 \otimes 1)(v' \otimes v) = 1(v') \otimes 1(v)$$
$$= v' \otimes v$$

$1 \otimes 1$ is id on $V' \otimes V$

b) Now let $V \xrightarrow{f} W \xrightarrow{g} Z$ be linear maps

have the two maps

i) $V' \otimes V \xrightarrow{1 \otimes f} V' \otimes W \xrightarrow{1 \otimes g} V' \otimes Z$

ii) $V' \otimes V \xrightarrow{1 \otimes g \circ f} V' \otimes Z$

$\Rightarrow$ The prop said there's a ! such map so $\underline{1 \otimes g \circ f} = \underline{1 \otimes g} \circ \underline{1 \otimes f}$

Rmk: This tells us the $\otimes$ is a "functor"!